LOCALLY COMPACT TRANSFORMATION GROUPS[1]

BY

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In §1 of this paper it is shown that a variety of conditions implying nice behavior for topological transformation groups are, in the presence of separability, equivalent. In §2 the continuity properties of the stability subgroups are studied. The conditions of §1 exclude the line acting on the torus in such a way that each orbit is dense. They exclude the integers acting on the circle by rotation through multiples of an irrational angle and they exclude the group of those sequences of zeros and ones which have all but a finite number of their terms equal to zero when this group acts on the space of all sequences of zeros and ones by coordinatewise addition (mod 2). As we shall see in the proof of Theorem 1, the latter transformation group is a prototype for all excluded transformation groups. This is analogous to the following fact in the theory of Rings of Operators: Every factor of type II_1 contains a hyperfinite factor of type II_1. The conditions were suggested by [3, Theorem 1] and the proof of their equivalence is somewhat analogous to the proof of [3, Theorem 1]. However, the proof does not depend upon [3] nor upon the theory of C^*-algebras.

THEOREM 1. *Let G be a locally compact Hausdorff topological transformation group acting on a locally compact space M. (By locally compact we mean that each neighborhood of a point in M contains a compact neighborhood of the point.) Suppose that the topologies of G and M have countable bases and that each nonempty locally compact subspace of M contains a nonempty relatively open Hausdorff subset. Then the following are equivalent*:

(1) *Each orbit in M is relatively open in its closure.*

(2) *M/G is T_0.*

(3) *M/G is countably separated.*

(4) *For each quasi-invariant ergodic Borel measure β, there is an orbit Gm in M such that $\beta(M\sim Gm)=0$.*

(5) *There is an ordinal γ and an ascending family $\{U_\alpha\}$ of open subsets of M/G indexed by the set of all ordinals less than or equal to γ such that $U_0=\emptyset$, $U_\gamma=M/G$, if α is a limit ordinal then $U_\alpha=\bigcup_{\alpha>\beta}U_\beta$ and if α is not a limit ordinal and not equal to 0 then $U_\alpha\sim U_{\alpha-1}$ is an open dense Hausdorff subset of $(M/G)\sim U_{\alpha-1}$.*

(6) *For each m in M, the map $gG_m\to gm$ from G/G_m onto Gm is a homeomorphism, where Gm has the relative topology as a subspace of M.*

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(7) *For each neighborhood N of e (the identity of G), each nonempty locally compact G-invariant subspace V of M and each nonempty relatively open subset V_0 of V there is a nonempty relatively open subset U of V_0 such that for each m in U, $Nm \cap U = Gm \cap U$.*

The hypotheses of Theorem 1 which concern M are satisfied when M is locally compact and Hausdorff and has a countable base for its topology. It may be useful to apply Theorem 1 to the case where M is the structure space of a *GCR* C^*-algebra (for a definition of *GCR*, see [5]). Such an M need not be Hausdorff or even T_1, but it is T_0, locally compact [2], and every nonempty locally compact subspace V contains a nonempty relatively open Hausdorff subset. When M is locally compact, this latter property is equivalent to the following: There is an ordinal γ and an ascending family $\{U_\alpha : \alpha \leqq \gamma\}$ of open subsets of M such that $U_0 = \varnothing$, $U_\gamma = M$, if α is a limit ordinal then $U_\alpha = \bigcup_{\alpha > \beta} U_\beta$ and if α is nonzero and is not a limit ordinal then $U_\alpha \sim U_{\alpha-1}$ is a dense Hausdorff subset of $M \sim U_{\alpha-1}$ (cf. [5, Theorem 6.2]). In fact if we are given the U_α's as above and if V is a nonempty locally compact subspace of M and if α is the smallest ordinal such that $V \cap U_\alpha \neq \varnothing$ then α is nonzero and is not a limit ordinal, and so $V \cap U_\alpha$ is a relatively open Hausdorff subset of V. Conversely suppose that each nonempty locally compact V contained in M contains a nonempty open Hausdorff subset. Suppose inductively that U_β has been chosen for all $\beta < \alpha$. If α is a limit ordinal, let $U_\alpha = \bigcup_{\alpha > \beta} U_\beta$. If $\alpha = 0$, let $U_0 = \varnothing$. If α is not a limit ordinal and is not zero, let U_α be chosen by Zorn's Lemma to be an open subset of M maximal with respect to containing $U_{\alpha-1}$ and having $U_\alpha \sim U_{\alpha-1}$ Hausdorff. Let V be the interior of $M \sim U_\alpha$ taken relative to $M \sim U_{\alpha-1}$. If $V \neq \varnothing$ then there is a contradiction involving the maximality of U_α. Thus $V = \varnothing$ and $U_\alpha \sim U_{\alpha-1}$ is dense in $M \sim U_{\alpha-1}$. If a space M satisfies these equivalent properties then we will say that M is *almost Hausdorff*. Returning to the case where M is the structure space of a C^*-algebra, M has a countable base for its topology if the C^*-algebra is separable.

If M is Hausdorff it is possible to associate with the action of G on M one (or several) C^*-algebras. These C^*-algebras are the completions, in suitable norms, of the algebra L defined in [1, p. 310]. If the algebras are *GCR* then the above conditions are satisfied. Condition (5) is then motivated by the following two facts: The structure space of the C^*-algebra is related to (and in general is somewhat more complicated than) the orbit space M/G; the structure space is almost Hausdorff if the C^*-algebra is *GCR*. Condition (7) states (roughly) that neighborhoods of e are just as transitive locally as G is.

We suppose that each g in G gives rise to a homeomorphism $m \to gm$ of M, that the map $g \to (m \to gm)$ is a homomorphism of G into the group of homeomorphisms of M and that the map $(g, m) \to gm$ from $G \times M$ onto M is continuous. $G_m = \{g : gm = m\}$ is the stability subgroup of m; e is the identity of G. The space M/G of orbits is endowed with the quotient topology. Let π

be the map from M onto M/G which sends an m in M onto the orbit Gm which contains it. We give M/G a Borel structure as follows: E is a Borel subset of M/G if $E=\pi(B)$ where B is a G-invariant Borel subset of M; the Borel subsets of M are the elements of the smallest σ-field containing the open sets. M/G is *countably separated* if there is a sequence $E_1, E_2, \cdots$ of Borel subsets of M/G which separate points of M/G (cf. [6]). A measure ν is quasi-invariant if G preserves sets of ν-measure zero. It is ergodic if it is quasi-invariant and if G acting on the measure space (M, ν) is ergodic, that is if G acting on the set of Borel sets modulo the Borel sets of ν-measure zero has no fixed points except for the equivalence classes of M and $\emptyset$.

1. **Proof of Theorem 1.** (1)$\Rightarrow$(2): If (1) is satisfied and if Gm_1 and Gm_2 are two distinct points of M/G, either $Gm_1^- \supset Gm_2$ or not. In the second case $Gm_1^- \cap Gm_2 = \emptyset$ and $M/G \sim \pi(Gm_1^-)$ is an open set containing Gm_2 but not Gm_1. In the first case there is by assumption an open set U in M containing Gm_1 but disjoint from Gm_2. $V=\bigcup\{gU: g\in G\}$ has the same properties as U and so we can suppose U is G-invariant. Then $\pi(U)$ is an open subset of M/G which contains Gm_1 but not Gm_2, and M/G is T_0.

(2)$\Rightarrow$(3): Suppose M/G is T_0 and let $U_1, U_2, \cdots$ be a base for open subsets of M. Then $E_1, E_2, \cdots$ is a base for open subsets of M/G, where $E_i=\pi(\bigcup\{gU: g\in G\})$. The sets E_i are Borel sets and they separate points of M/G since M/G is T_0.

(3)$\Rightarrow$(4) (cf. [6]): Let $E_1, E_2, \cdots$ be a sequence of Borel sets which separate points of M/G, let $F_i=\pi^{-1}(E_i)$, let $H=\bigcap_i H_i$, where $H_i=F_i$ if $\beta(M\sim F_i)=0$, $H_i=M\sim F_i$ if $\beta(F_i)=0$. Then H is an orbit and $\beta(M\sim H) \leqq \sum_i \beta(M\sim H_i)=0$.

We introduce a new condition:

(8) *For each nonempty locally compact G-invariant subset V of M and each relatively open nonempty subset V_0 of V there is a compact neighborhood K of e and a relatively open nonempty subset U of V_0 such that if $g\in G$, if U_0 is a relatively open nonempty subset of U and if $gU_0\subset U$ then $KU_0\cap gU_0\neq\emptyset$.*

(4)$\Rightarrow$(8): We assume the denial of (8) and prove the denial of (4). That is we assume that there is a nonempty locally compact G-invariant subspace V of M and a nonempty relatively open subset V_0 of V such that for each compact neighborhood K of e and each relatively open nonempty subset U of V_0 there is a g in G and a relatively open nonempty subset U_0 of U such that $gU_0\subset U$ and $KU_0\cap gU_0=\emptyset$.

We observe that a locally compact subset E of M is a Borel set. In fact if the family $\{U_\alpha: \alpha\leqq\gamma\}$ is as in the paragraph following the statement of Theorem 1 then γ must be countable, since M has a countable base for open sets. For any ordinal α, the set $(E\cap U_{\alpha+1})\sim U_\alpha$ is locally compact and so is a countable union of compact (and therefore relatively closed) subsets of $U_{\alpha+1}\sim U_\alpha$. Thus it is a Borel set, and so is

$$E = \bigcup_{\gamma \geq \alpha} (E \cap U_{\alpha+1}) \sim U_\alpha.$$

It is no loss of generality to suppose that $V=M$ and that V_0 is Hausdorff. Let N be a compact symmetric neighborhood of e; let $W_1, W_2, \cdots$ be a base for open sets in M. We choose compact subsets $U(i_1, \cdots, i_n)$ of M with nonempty interiors and elements $g(n)$ of G, where $i_k=0$ or 1 and $n=0, 1, \cdots$, which satisfy

(1.1) $U(i_1, \cdots, i_{r-1}, i_r) \subset U(i_1, \cdots, i_{r-1}),$

(1.2) $g(s)U(0_s, i_{s+1}, \cdots, i_r) = U(0_{s-1}, 1, i_{s+1}, \cdots, i_r);$ $\quad 1 \leqq s < r,$

(1.3) $NU(i_1, \cdots, i_r) \cap U(j_1, \cdots, j_r) = \emptyset$ unless $i_1 = j_1, \cdots, i_r = j_r,$

(1.4) if $j \leqq r$ then either:

$$U(i_1, \cdots, i_r) \subset W_j \quad \text{or} \quad U(i_1, \cdots, i_r) \cap W_j = \emptyset$$

if $r \geqq 1$, where 0_r is the family of r zeros.

Let $U(\emptyset)$ be a compact subset of V_0 with a nonempty interior, let $g(0)=e$ and suppose inductively that n is a non-negative integer and that $U(i_1, \cdots, i_r)$ and $g(r)$ have been chosen for $r = 0, \cdots, n$ and that (1.1), $\cdots$, (1.4) are satisfied if $r=1, \cdots, n$ and that $U(0_n)$ has a nonempty interior. If $n=0$ the inductive hypothesis is true. If $n>0$, let

$$K = \bigcup_{1 \leq r_1 < \cdots < r_s \leq n} g(r_s)^{-1} \cdots g(r_1)^{-1} N g(r_1) \cdots g(r_s)$$

where the union is taken over all monotone increasing ordered subsets $\{r_1, \cdots, r_s\}$ of $\{1, \cdots, n\}$. If $n=0$, let $K=N$. By assumption there is an open nonempty subset U_0 of Int $U(0_n)$ and a g in G such that $gU_0 \subset \text{Int } U(0_n)$ and $KU_0 \cap gU_0 = \emptyset$. Let $g(n+1)=g$, let $U(0_{n+1})$ be a compact subset of U_0 with a nonempty interior and chosen so that if $1 \leqq j \leqq n+1$ and if $\{r_1, \cdots, r_s\}$ is any monotone increasing ordered subset of $\{1, \cdots, n+1\}$ then either $g(r_1) \cdots g(r_s)U(0_{n+1}) \subset W_j$ or $g(r_1) \cdots g(r_s)U(0_{n+1}) \cap W_j = \emptyset$.

Let $\{i_1, \cdots, i_{n+1}\}$ be a sequence of zeros and ones, let k_r be the position of the rth 1 in this sequence. Let $U(i_1, \cdots, i_{n+1}) = g(k_1) \cdots g(k_t)U(0_{n+1})$ where t is the number of ones in $\{i_1, \cdots, i_{n+1}\}$. Then (1.4) is satisfied for $r=n+1$. If $k_t \neq n+1$ then (1.2) implies that

$$(1.5) \qquad U(i_1, \cdots, i_n) = g(k_1) \cdots g(k_t)U(0_n)$$

and since $U(0_{n+1}), g(n+1)U(0_{n+1}) \subset U(0_n)$, (1.1) is satisfied for $r=n+1$. If $i_1, \cdots, i_s = 0_s$ then

$$\begin{aligned} g(s)U(0_s, i_{s+1}, \cdots, i_{n+1}) &= g(s)g(k_1) \cdots g(k_t)U(0_{n+1}) \\ &= U(0_{s-1}, 1, i_{s+1}, \cdots, i_{n+1}) \end{aligned}$$

and (1.2) is satisfied for $r=n+1$. It suffices to prove (1.3) when $i_1=j_1, \cdots, i_n=j_n$, $i_{n+1}=0$ and $j_{n+1}=1$ (since $N=N^{-1}$). Then

$$U(i_1, \cdots, i_n, 0) = g(k_1) \cdots g(k_s)U(0_{n+1}),$$
$$U(j_1, \cdots, j_n, 1) = g(k_1) \cdots g(k_s)g(n+1)U(0_{n+1}),$$

and since

$$KU(0_{n+1}) \cap g(n+1)U(0_{n+1}) = \emptyset,$$

we have

$$Ng(k_1) \cdots g(k_s)U(0_{n+1}) \cap g(k_1) \cdots g(k_s)g(n+1)U(0_{n+1}) = \emptyset$$

and (1.3) is proved for $r=n+1$. This completes the induction and so the U's and g's can be constructed.

Let

$$V(n) = \bigcup \{U(i_1, \cdots, i_n): i_1 = 0 \text{ or } 1, \cdots, i_n = 0 \text{ or } 1\},$$

let $C=\bigcap_n V(n)$, let $C(i_1, \cdots, i_n)=C\cap U(i_1, \cdots, i_n)$. Let X be the set of sequences of zeros and ones with the topology of pointwise convergence; we construct a homeomorphism Φ of C onto X. If $\{i_n\}\in X$ then $\bigcap_n C(i_1, \cdots, i_n)$ contains exactly one element, $c(\{i_n\})$. In fact the sets $C(i_1, \cdots, i_n)$ are closed relative to the compact set $U(\emptyset)$ (since $U(\emptyset)$ is Hausdorff) and these sets have the finite intersection property. Furthermore, if c and c_0 are in $\bigcap_n C(i_1, \cdots, i_n)$ then by (1.4), c and c_0 are not separated by W_j for any j. Since M is T_0 and $W_1, W_2, \cdots$ is a base, $c=c_0$. Let $\Phi(c(\{i_n\}))=i_n$; Φ is a one-one map of C onto X. The inverse image of the basic open set $\{\{i_n\}: i_1=a_1, \cdots, i_k=a_k\}$ is $C(a_1, \cdots, a_k)$, which is relatively open in C. Thus Φ is continuous, and Φ is a homeomorphism since C is compact and X is Hausdorff. The set of intersections with C of the Borel subsets of M is the set of Borel subsets of C and this is the set of inverse images under Φ of the Borel subsets of X. Thus we can define a unique Borel measure λ on M by the formulas

$$\lambda(C(i_1, \cdots, i_n)) = 2^{-n}$$
$$\lambda(M \sim C) = 0.$$

Let ν be a finite measure on G, equivalent to Haar measure, and if B is a Borel subset of M, let

$$\beta(B) = \int \lambda(hB)d\nu(h). \tag{1.6}$$

We show that the integral in (1.6) exists. Let U denote the interior of $U(\emptyset)$. Since C is compact and $C\subset U$ there is a symmetric open neighborhood P of e such that $PC\subset U$. If f is a real valued function defined on U and if f is continuous and has compact support relative to U then the function

$$h \to f(h\cdot) \mid C$$

which maps h onto the restriction $f(h\cdot)|C$ to C of the function $m \to f(hm)$ is a continuous function of h, in the topology of uniform convergence for functions on C, provided $h \in P$. Thus

$$h \to \int f(hm)d\lambda(m)$$

is a continuous function of h, for $h \in P$.

Let K be a compact subset of U, let χ_K be its characteristic function, let χ_P be the characteristic function of P. There is a monotonically decreasing sequence f_n of continuous functions defined on U and with compact support relative to U which converges to χ_K pointwise. Then

$$\chi_P(h)\lambda(hK) = \chi_P(h)\int \chi_{hK}(m)d\lambda(m) = \chi_P(h)\int \chi_K(h^{-1}m)d\lambda(m)$$
$$= \chi_P(h)\lim_n \int f_n(h^{-1}m)d\lambda(m)$$

and so $\chi_P(h)\lambda(hK)$ is a measurable function of h. Let $\mathcal{S}$ be the set of Borel subsets B of M for which $\chi_P(h)\lambda(hB)$ is a measurable function of h. Then $\mathcal{S}$ contains the compact subsets of U and since there is a sequence of compact subsets of U whose union is U, $\mathcal{S}$ contains the relatively closed subsets of U. Since $\mathcal{S}$ is closed under monotone limits, $\mathcal{S}$ contains the Borel subsets of U and since $\mathcal{S}$ contains any Borel set disjoint from U, $\mathcal{S}$ is the set of Borel subsets of M. There is a sequence $g_1, g_2, \cdots$ of elements of G and a sequence $P_1, P_2, \cdots$ of Borel subsets of P such that the sets $P_1g_1, P_2g_2, \cdots$ are disjoint and such that their union is G. If B is a Borel subset of M then

$$\lambda(hB) = \sum_{i=1}^{\infty} \chi_{P_ig_i}(h)\lambda(hB)$$
$$= \sum_{i=1}^{\infty} \chi_{P_i}(hg_i^{-1})\lambda(hg_i^{-1}g_iB),$$

which is a convergent sum of translates of the measurable functions $\chi_{P_i}(h)\lambda(hg_iB)$ and so is measurable. Since $0 \leqq \lambda(hB) \leqq 1$, the integral in (1.6) exists, and is non-negative and finite, and it is easy to see that the integral in (1.6) defines a nonzero finite Borel measure β on M.

β is quasi-invariant since $\beta(B)=0$ iff $\lambda(hB)=0$ for a.e. h iff $\lambda(hgB)=0$ for a.e. h iff $\beta(gB)=0$. There is a sequence $h_1, h_2, \cdots$ such that $N_1h_1, N_1h_2, \cdots$ is a cover for G, where N_1 is some neighborhood of e such that $N_1(N_1)^{-1} \subset N$. If m is in M and if $n_1h_im, n_2h_im \in C$ for some n_1 and n_2 in N_1 then n_1h_im, n_2h_im belong to the same sets $U(i_1, \cdots, i_r)$ by (1.3) and are equal by (1.4). Thus $Gm \cap C$ is countable and so $\lambda(hGm) = \lambda(Gm) = 0$ and $\beta(Gm) = 0$. Hence there is no orbit Gm for which $\beta(M \sim Gm) = 0$.

We show that G acts ergodicly on M with resepect to β, and this will complete the proof of (4)$\Rightarrow$(8). Let R be a Borel subset of M such that $\beta(R\Delta gR)=0$ for all g in G. Let

$$a(\emptyset)(h) = \lambda(hR)$$
$$a(i_1, \cdots, i_n)(h) = \lambda(hR \cap U(i_1, \cdots, i_n)).$$

Then

$$\begin{aligned} &| a(i_1, \cdots, i_n)(h) - a(i_1, \cdots, i_n)(hg) | \\ &\quad = | \lambda(hR \cap U(i_1, \cdots, i_n)) - \lambda(hgR \cap U(i_1, \cdots, i_n)) | \\ &\quad \leqq \lambda([hR \cap U(i_1, \cdots, i_n)]\Delta[hgR \cap U(u_1, \cdots, i_n)]) \\ &\quad = \lambda([hR\Delta hgR] \cap U(i_1, \cdots, i_n)) \leqq \lambda(hR\Delta hgR) = 0 \qquad \text{for a.e. } h, \end{aligned}$$

and so $a(i_1, \cdots, i_n)(\cdot) = a(i_1, \cdots, i_n)(\cdot g)$ a.e. Since G acting on itself by right translations acts ergodicly, $a(i_1, \cdots, i_n)(\cdot)$ is equal a.e. to some constant $A(i_1, \cdots, i_n)$, and the same proof shows that $a(\emptyset)(\cdot)$ is equal a.e. to a constant $A(\emptyset)$.

We let $g = g(k_1) \cdots g(k_t)$ where $k_1, \cdots, k_t$ are chosen as in (1.5), and it then follows from (1.2) that

$$U(i_n, \cdots, i_n, i_{n+1}, \cdots, i_{n+p}) = gU(0_n, i_{n+1}, \cdots, i_{n+p})$$

and so

$$\lambda(B \cap U(0_n)) = \lambda(gB \cap U(i_1, \cdots, i_n))$$

for all Borel sets B. For a.e. h,

$$A(0_n) = \lambda(hR \cap U(0_n)) = \lambda(ghR \cap U(i_1, \cdots, i_n)) = A(i_1, \cdots, i_n).$$

Since $A(\emptyset) = \sum_{i_1, \cdots, i_n} A(i_1, \cdots, i_n)$ for each n, $A(i_1, \cdots, i_n) = 2^{-n}A(\emptyset)$ and it follows that there is a subset T of G of measure zero such that if $h \notin T$, then

$$\lambda(hR \cap U(i_1, \cdots, i_n)) = 2^{-n}\lambda(hR).$$

It is well known and easy to prove that this implies that for $h \notin T$, $\lambda(hR) \in \{0, 1\}$ (cf. [4, p. 201, problem (3)]). Thus either $\lambda(hR) = 0$ for a.e. h or $\lambda(h(M \sim R)) = 0$ for a.e. h and either $\beta(R) = 0$ or $\beta(M \sim R) = 0$, and β is ergodic.

We remark that since β is finite we have actually proved (4′)$\Rightarrow$(8) where (4′) is the condition obtained from (4) by adding "finite" in front of "Borel measure" in (4).

(8)$\Rightarrow$(6): We assume the denial of (6) and we prove the denial of (8). That is, we assume that there is an m in M such that the map θ given in (6) is not a homeomorphism. Let $V = Gm^-$; it is no loss of generality to suppose that $Gm^- = M$. Let $V_0 = M$, let U be a nonempty open subset of M, let K be a compact neighborhood of e and let p be in $U\cap Gm$. Since M is almost Hausdorff, M contains a dense open Hausdorff subset. This subset meets Gm and so each point in Gm has a dense open Hausdorff neighborhood. Since Kp is compact, there is a finite number $W_1, \cdots, W_n$ of dense open Hausdorff subsets of M which form a cover for Kp; their intersection W is dense and open. Thus $W\cap U\neq\emptyset$; since this set is open, $W\cap U\cap Gm\neq\emptyset$. Let $q\in W\cap U\cap Gm$. Since θ^{-1} is not continuous at q, there is a sequence $g_1, g_2, \cdots$ in G such that $g_im\in W\cap U$ and $g_im\rightarrow q$ but g_iG_m not $\rightarrow h_2G_m$, where h_2 is an element of G such that $h_2m=q$. Choose h_1 in G so that $h_1m=p$. Suppose $g_im\in Kp$ for all i. Then $g_iG_m\in Kh_1G_m$ and so the sequence $\{g_iG_m\}$ has a limit point kh_1G_m in $Kh_1G_m\sim\{h_2G_m\}$ for some $k\in K$. Thus $\{\theta(g_iG_m)\}=\{g_im\}$ has a limit point kp in $Kp\sim\{q\}$, and so there is a subsequence of $\{g_im\}$ converging to the distinct points q and kp. However, for some j, q and kp are in the same Hausdorff open set W_j, a contradiction, and so $g_im\notin Kp$ for some i. Choosing such an i, we assert that Kp and g_im have neighborhoods V_1 and V_2 respectively which are disjoint. In fact if $kp\in Kp$ then $kp\in W_j$ for some j and since W_j is Hausdorff, kp and g_im have open neighborhoods V_{kp1} and V_{kp2} respectively which are disjoint. By the compactness of Kp, there is a finite subset K_0 of K for which the set $V_1=\bigcup\{V_{kp1}: k\in K_0\}$ contains Kp. V_1 is a neighborhood of Kp and it is disjoint from the set $V_2=\bigcap\{V_{kp2}: k\in K_0\}$, which is a neighborhood of g_im. There is an open neighborhood U_0 of p contained in U such that $KU_0\subset V_1$, $g_ih_1^{-1}U_0\subset V_2$ and $g_ih_1^{-1}U_0\subset U$. (Observe that $g_im=g_ih_1^{-1}p$). We let $g=g_ih_1^{-1}$; then $KU_0\cap gU_0\subset V_2\cap V_1=\emptyset$ which contradicts (8) and so (8)$\Rightarrow$(6).

We introduce a new condition:

(9) *For each neighborhood N of e, each nonempty locally compact G-invariant subspace V of M and each nonempty relatively open subset V_0 of V there is a nonempty relatively open subset U of V_0 such that if $g\in G$ and if U_0 is a nonempty relatively open subset of U such that $gU_0\subset U$ then $NU_0\cap gU_0\neq\emptyset$.*

(6)$\Rightarrow$(9): We assume the denial of (9) and prove the denial of (6). That is, we assume that there is a neighborhood N of e, a nonempty locally compact G-invariant subspace V of M and a nonempty relatively open subset V_0 of V such that if U is a nonempty relatively open subset of V_0 then there is a g in G and a nonempty relatively open subset U_0 of U such that $gU_0\subset U$ and

$$NU_0\cap gU_0=\emptyset. \tag{1.7}$$

It is no loss of generality to suppose that $V=M$ and that V_0 is Hausdorff. We choose by induction compact subsets $E(n)$ of V_0 with nonempty interiors and elements g_n of G for $n=0, 1, 2, \cdots$ such that if $W_1, W_2, \cdots$ are a base for the topology of M then

(1.8) $g_{n+1}E(n+1)$, $E(n+1)\subset E(n)$, $n=0, 1, \cdots$.

(1.9) $NE(n)\cap g_nE(n)=\emptyset$, $n=1, 2, \cdots$.

(1.10) If $n \geqq j$ then either: $E(n) \subset W_j$ or $E(n) \cap W_j = \emptyset$.
Let $E(0)$ be a compact subset of V_0 with a nonempty interior, let $g_0 = e$. If $E(n)$ and g_n have been chosen for some $n \geqq 0$, let U be the interior of $E(n)$ and choose a $g = g_{n+1}$ in G and a nonempty open subset U_0 of U such that $g_{n+1}U_0 \subset U$ and (1.7) is satisfied. Let $E(n+1)$ be a compact subset of U_0 with a nonempty interior and such that (1.10) is satisfied for $n+1$. Then (1.8) and (1.9) are satisfied for $n+1$.

Since $E(n)$ is relatively closed in $E(0)$ and since $E(0)$ is compact, there is an m in $\bigcap_n E(n)$. It follows from (1.10) that $g_n m \to m$ and from (1.9) that $g_n m \notin Nm$ for any n. Thus Nm is not a neighborhood of m relative to Gm; the map given in (6) is not a homeomorphism.

(9)$\Rightarrow$(7): We may suppose $V = M$. Let N and V_0 be given as in (7). Choose a compact neighborhood N_1 of e contained in N and nonempty open Hausdorff subsets W_1 and W_2 of V_0 such that $N_1 W_1 \subset W_2$. Let U be chosen by (9) applied to the neighborhood N_1 and the open subset W_1 of $V = M$. Let m be in U, let $U_1, U_2, \cdots$ be a decreasing open basis for neighborhoods of m and let gm be in $Gm \cap U$. For large i, $U_i \subset U$ and $gU_i \subset U$ and so there are m_i and n_i in U_i and an h_i in N_1 such that $h_i m_i = g n_i$. There is a subsequence $h_{i(1)}, h_{i(2)}, \cdots$ of $h_1, h_2, \cdots$ such that $h_{i(k)}$ tends to some h in N_1. Since $hm \in N_1 W_1 \subset W_2$ and since W_2 is Hausdorff,

$$hm = \lim_k h_{i(k)} m_{i(k)} = \lim_k g n_{i(k)} = gm,$$

and so $Nm \cap U \supset Gm \cap U$; $Nm \cap U = Gm \cap U$.

(7)$\Rightarrow$(5): As in the paragraph following the statement of Theorem 1, it is sufficient to prove: For each nonempty locally compact G-invariant subspace V of M there is a relatively open nonempty G-invariant subset W of V such that $\pi(W)$ is Hausdorff. It is sufficient to prove this when $V = M$, and assuming this let V_0 and V_1 be nonempty open Hausdorff subsets of M and let N be a compact neighborhood of e and let V_0, V_1 and N be chosen so that $NV_0 \subset V_1$. Let U be chosen by (7) and let $W = \bigcup \{gU : g \in G\}$. Let Gm_i be a sequence of orbits in $\pi(W)$ converging to Gs and Gt, where Gs and $Gt \in \pi(W)$. We must prove that $Gs = Gt$; we can suppose $s, t \in U$. Let $S_1, S_2, \cdots$ (resp. $T_1, T_2, \cdots$) be a base of open neighborhoods of s (resp. t) such that $S_k \subset U$ and $T_k \subset U$ for all k. If k is chosen and if i is sufficiently large then $Gm_i \subset \pi^{-1}\pi(S_k)$ and $Gm_i \subset \pi^{-1}\pi(T_k)$. Thus there is an $m(k)$ in U such that $m(k) \in S_k$ and a $g(k)$ in G such that $g(k)m(k) \in T_k$. By (7) there is an $n(k)$ in N such that $n(k)m(k) = g(k)m(k)$. Passing to a subsequence of the k's if necessary, we can suppose that $n(k) \to n$ for some n in N. Then

$$\lim_k n(k)m(k) = t,$$

$$\lim_k n(k)m(k) = \lim_k n(k) \lim_k m(k) = ns.$$

Since t and ns both belong to the open Hausdorff set V_1, this proves that $t=ns$, $Gs=Gt$, and so $\pi(W)$ is Hausdorff.

(5)$\Rightarrow$(1): If $m\in M$, there is a smallest α such that $Gm\subset U_\alpha$. Since Gm^- is G-invariant, $Gm^-=\pi^{-1}(\{Gm\}^-)$. (Observe that $\{Gm\}$ is a subset of G/M.) Since $\{Gm\}\in(M/G)\sim U_{\alpha-1}$, $\{Gm\}^-\subset(M/G)\sim U_{\alpha-1}$ (α cannot be a limit ordinal). Since $U_\alpha\sim U_{\alpha-1}$ is Hausdorff, $\{Gm\}^-\cap(U_\alpha\sim U_{\alpha-1})=\{Gm\}$. Thus $\{Gm\}^-\cap U_\alpha=\{Gm\}$, $Gm^-\cap\pi^{-1}(U_\alpha)=Gm$ and Gm is relatively open in Gm^-. This completes the proof Theorem of 1.

The proof of (8)$\Rightarrow$(6) raises the question: If G and M satisfy the hypotheses of Theorem 1 and if $m\in M$, is Gm Hausdorff? If (6) is satisfied the answer is yes, since G/G_m is Hausdorff. In any case Gm is T_1.

It is now easy to see that (3′) is equivalent to the previous conditions, where (3′) is the following:

(3′) *If β is a finite Borel measure on M then there are G-invariant Borel subsets $N, E_1, E_2, \cdots$ of M such that $\beta(N)=0$ and $\pi(E_1), \pi(E_2), \cdots$ separate points of $M/G\sim\pi(N)$.*

In fact (3)$\Rightarrow$(3′), it is easy to see that (3′)$\Rightarrow$(4′), and we have already observed that (4′)$\Rightarrow$(8). If σ is a multiplier for G [7, p. 267], if K is a closed normal subgroup of G and if $\hat{K}^\sigma$ (defined in [7, p. 272]) is type I then $\hat{K}^\sigma$ and the natural action of G on $\hat{K}^\sigma$ satisfy the hypothesis of Theorem 1. (3′) in the case $M=\hat{K}^\sigma$ is the statement that K is σ-regularly imbedded in G [7, p. 302], and so Theorem 1 provides a number of properties equivalent to K being σ-regularly imbedded.

2. **The stability subgroups.** Throughout this section we suppose that M is Hausdorff. Let m be in M, let N be a neighborhood of the identity of G. Then we say that the *stability groups are continuous at m* if for every sequence $\{m_i\}$ in M converging to m and for each g in G_m there is a sequence $\{g_i\}$ such that $g_i\in G_{m_i}$ and $g_i\to g$; we say that the *stability groups jump by no more than N at m* if for each sequence $\{m_i\}$ in M converging to m and for each g in G_m there is a sequence $\{g_i\}$ such that $g_i\in NG_{m_i}$ and $g_i\to g$. The conditions of Theorem 1 are neither necessary nor sufficient for the continuity of the stability groups at each m in M. Each of the transformation groups mentioned in the introduction has the property that $G_m=\{e\}$ for each m, and so for these groups the stability groups are continuous. On the other hand if G is the circle group acting by rotation on the plane M then G and M satisfy the conditions of Theorem 1 but the stability groups are discontinuous at the origin. If the conditions of Theorem 1 are satisfied then we prove that there is some continuity in the stability subgroups.

THEOREM 2. *Let G and M satisfy the hypothesis and the conditions* (1), $\cdots$, (7) *of Theorem* 1, *let M be Hausdorff. Let N be a neighborhood of the identity of G. There is an open dense subset U of M such that if $m\in U$ then the stability groups jump by no more than N at m. There is a subset P of M such that $M\sim P$*

is of the first category in M and such that if $m\in P$ then the stability groups are continuous at m.

Let U be the union of the open subsets of M upon which the stability groups jump by no more than N. The stability groups jump by no more than N at the points of U. If U is not dense then $V_0=M\sim(U^-)$ is a nonempty open subspace of M. Let U_1 be the U chosen by condition (7) of Theorem 1 in the case $V=M$. Let m be in U_1, let $\{m_i\}$ be a sequence in U_1 converging to m, let g be in G_m. Since $gm=m\in U_1$, $gm_i\in U_1$ for all sufficiently large i. Thus $n_im_i=gm_i$, $m_i=n_i^{-1}gm_i$ and $n_i^{-1}g\in G_{m_i}$ for some n_i in N. Hence $g\in NG_{m_i}$ for large i and so the stability groups do not jump by more than N at the points of U_1 and also at the points of $U\cup U_1$. This contradicts the choice of U and so U is dense in M.

Let $N_1, N_2, \cdots$ be a base for neighborhoods of e, let U_i be an open dense subset of M upon which the stability groups have jumps no larger than N_i, and let $P=\bigcap_i U_i$. Then $M\sim P$ is of the first category. Let m be in P, let $\{m_k\}$ be a sequence in M converging to m and let g be in G_m. For each k let $i=i(k)$ be the largest integer less than k and such that if $p\geq k$ then $g\in N_i^{-1}N_iG_{m_p}$, if such an i exists; if not, let $i(k)=0$. If s is a positive integer then m_k is eventually in U_s and $g=\lim h_k$ where $h_k\in N_sG_{m_k}$. Since h_k is eventually in N_sg, $g\in N_s^{-1}N_sG_{m_k}$ for large k, and so $i(k)\geq s$ for large k. For each k choose a g_k in G_{m_k} and an n_k in $N_{i(k)}^{-1}N_{i(k)}$ such that $g=n_kg_k$. Since $i(k)\to\infty$, $n_k\to e$ and $g=\lim g_k$. This proves that the stability groups are continuous at m.

We show that it may not be possible to choose P open. Let G be an infinite direct product of groups $\{0, 1\}$, let M be an infinite direct product of intervals $[-1, 1]$. Let the action of G on M be given by each factor $\{0, 1\}$ of G acting by reflection about the origin on the corresponding factor $[-1, 1]$ of M. Let $m=(x_1, x_2, \cdots)$ be in M. Then $G_m=\{e\}$ if and only if $x_i\neq 0$ for all i. If $P=\{m=(x_1, x_2, \cdots): x_i\neq 0 \text{ for all } i\}$ then it follows that the stability groups are continuous at m if and only if $m\in P$. Since the complement of P is dense, the only open set upon which the stability groups are continuous is the empty set. The G of this example is not a Lie group.

THEOREM 3. *Let G and M satisfy the hypothesis and the conditions* (1), $\cdots$, (7) *of Theorem* 1 *and suppose that M is Hausdorff and G is a Lie group. Then there is a dense open subset P of M upon which the stability groups are continuous.*

Let P be the union of the open sets upon which the stability groups are continuous. Then P is open and G-invariant and the stability groups are continuous on P. To prove that P is dense we assume the contrary and consider the action of G on $\operatorname{Int}(M\sim P)$ (which, by our assumption, is nonempty). It suffices to prove that $\operatorname{Int}(M\sim P)$ contains a nonempty open set upon which the stability groups are continuous, since this contradicts the choice of P.

Since the action of G on $\operatorname{Int}(M\sim P)$ satisfies the hypothesis of Theorem 3, it suffices to prove that under the hypothesis of Theorem 3, there is a nonempty open set upon which the stability groups are continuous.

Let G_{m0} be the identity component of G_m. We show that $\dim G_{m0}$ is an upper semicontinuous function of m. Let W be a neighborhood of zero in the Lie algebra $\mathfrak{g}$ of G such that $\exp \mid W$ is $1-1$. There is a basis $x_1, \cdots, x_n$ for $\mathfrak{g}$ such that if $|a_i| \leqq 1$ for $i=1, \cdots, n$, then $\sum_{i=1}^n a_i x_i \in W$. We choose an inner product in $\mathfrak{g}$ such that the $x_1, \cdots, x_n$ are orthonormal. Let $\mathfrak{g}_m$ be the Lie algebra of G_m, let $y_{m1}, \cdots, y_{ms(m)}$ be an orthonormal basis for $\mathfrak{g}_m$. Let b be a number, let $\{m_k\}$ be a sequence in $\{m: \dim G_{m0} \geqq b\}$ converging to some m_0 in M. Passing to a subsequence we can suppose that for $1 \leqq i \leqq \limsup_k s(m_k)$ the sequence $\{y_{m_k i}: k=a, a+1, a+2, \cdots\}$ converges to an element z_i of W, where a is a suitable positive integer. By continuity, the $z_1, \cdots, z_s$ are orthonormal and so linearly independent, where $s = \limsup_k s(m_k) = \limsup_k \dim G_{m_k 0}$. If $|\gamma| \leqq 1$ then

$$\exp(\gamma z_i) m_0 = \lim_k \exp(\gamma y_{m_k i}) m_k = \lim_k m_k = m_0$$

and so $\exp(\gamma z_i) \in G_{m_0}$. Thus $\exp(z_i) \in G_{m_0 0}$, $z_i \in \mathfrak{g}_{m_0}$ and

$$\dim G_{m_0 0} = \dim \mathfrak{G}_{m_0} \geqq s = \limsup_k \dim G_{m_k 0}, \tag{2.1}$$

so $m_0 \in \{m: \dim G_{m0} \geqq b\}$ and $\{m: \dim G_{m0} \geqq b\}$ is closed, which proves the upper semicontinuity. It follows that the set V upon which $\dim G_{m0}$ assumes its minimum is open; moreover, V is nonempty. We remark that if $m_0 \in V$ then there must be equality in (2.1) and so $\{z_1, \cdots, z_s\}$ is a basis for $\mathfrak{g}_{m_0}$. For convenient reference we state this explicitly: *If $m \in V$ and m_k is a sequence in V converging to m and if $\{y_{1k}, \cdots, y_{1s}\}$ is an orthonormal basis for $\mathfrak{g}_{m_k}$ then there is a subsequence $k(1), k(2), \cdots$ such that $\lim_p y_{jk(p)} = z_j$ exists for $1 \leqq j \leqq s$ and $\{z_1, \cdots, z_s\}$ is an orthonormal basis for $\mathfrak{g}_m$.*

Suppose we can find a nonempty open subset U_0 of V and a compact neighborhood N of e such that $G_{m0} \cap N = G_m \cap N$ for m in U_0. Let U be chosen by Theorem 2, let m be in $U \cap U_0$, let m_k be a sequence in M converging to m, let g be in G_m. By the choice of U, there are sequences n_k and g_k such that $n_k \in N$, $g_k \in G_{m_k}$ and $n_k g_k \to g$. Passing to a subsequence if necessary, $n_k \to n$ for some n in N. Thus $g_k \to n^{-1}g$, $n^{-1}g \in G_m$ and so $n \in G_m \cap N$ and $n \in G_{m0}$. There are $w_1, \cdots, w_\nu$ in $\mathfrak{g}_m$ such that $n = \exp(w_1) \cdots \exp(w_\nu)$, since $\exp(\mathfrak{g}_m)$ contains a neighborhood of the identity in G_{m0} and any such neighborhood generates G_{m0} as a group. Applying the argument of the preceding paragraph and passing to a subsequence of the sequence $m_1, m_2, \cdots$

$$w_\mu = \lim_k w_{\mu k}, \qquad 1 \leqq \mu \leqq \nu,$$

where $w_{\mu k}$ is a suitable element of $\mathfrak{g}_{m_k}$, and so

$$g = \lim_k \exp(w_{1k}) \cdots \exp(w_{\nu k}) g_k.$$

We have proved that there is a subsequence $m_{k(1)}, m_{k(2)}, \cdots$ of $m_1, m_2, \cdots$ and there are g_i in $G_{m_{k(i)}}$ such that $g_i \to g$. The following lemma shows that the stability groups are continuous on $U \cap U_0$, and since $U \cap U_0 \neq \emptyset$, the proof of Theorem 3 will then be complete.

LEMMA 4. *Let G and M satisfy the hypotheses of Theorem* 1, *let M be Hausdorff and let m be in M. Then the following are equivalent*:

(1) *The stability groups are continuous at m.*

(2) *For each g in G_m and each neighborhood N of e there is a neighborhood U of m such that if $p \in U$ then $g \in NG_p$.*

(3) *If m_k is a sequence in M converging to m and if $g \in G_m$ then there is a subsequence $m_{k(1)}, m_{k(2)}, \cdots$ and there is g_i in $G_{m_{k(i)}}$ such that $g_i \to g$.*

We prove that such U_0 and N exist. If $T = \sum_i t_i x_i$ and $S = \sum_i s_i x_i$ are in a suitable neighborhood of zero in $\mathfrak{g}$ then $\exp(-S)\exp(S+T) = \exp(R)$ where $R = \sum_i r_i x_i$ and

$$r_i = r_i(s_1, \cdots, s_n, t_1, \cdots, t_n)$$

is an analytic function of s_j and t_k such that $r_i(0, \cdots, 0) = 0$. Also

$$\begin{aligned} r_i = r_{i0}(s_1, \cdots, s_n) &+ \sum_j t_j r_{ij}(s_1, \cdots, s_n) \\ &+ \sum_{j,k} t_j t_k r_{ijk}(s_1, \cdots, s_n, t_1, \cdots, t_n) \end{aligned} \tag{2.2}$$

where r_{i0}, r_{ij} and r_{ijk} are analytic functions. If $T = 0$ then $R = 0$,

$$0 = r_i(s_1, \cdots, s_n, 0, \cdots, 0) = r_{i0}(s_1, \cdots, s_n)$$

and so $r_{i0} \equiv 0$. If $S = 0$ then $R = T$ and so

$$r_{ij}(0, \cdots, 0) = \delta_{ij},$$
$$r_{ijk}(0, \cdots, 0, t_1, \cdots, t_n) = 0.$$

Thus we can find a positive ϵ such that the functions in (2.2) are defined and analytic and such that

$$|r_{ij}(s_1, \cdots, s_n) - \delta_{ij}| < 1/6n^3,$$
$$|r_{ijk}(s_1, \cdots, s_n, t_1, \cdots, t_n)| < 1/6n^3,$$

if $|s_\sigma|, |t_\tau| \leq \epsilon$ for $1 \leq \sigma, \tau \leq n$.

Let K be a compact subset of V with a nonempty interior U_0. Suppose that there is no neighborhood N of e such that for m in U_0, $G_{m0} \cap N = G_m \cap N$. Then there is a sequence m_q in U_0 converging to some m in K and a g_q in $G_{m_q} \sim G_{m_q 0}$ such that $g_q \to e$. For sufficiently large q we can write $g_q = \exp(Y_q)$ and we can write $Y_q = S_q + T_q$ where $S_q \in \mathfrak{g}_{m_q}$ and $T_q \perp \mathfrak{g}_{m_q}$. (Recall that $\mathfrak{g}$ has the inner product in which the family $\{x_1, \cdots, x_n\}$ is orthonormal.) We can choose Y_q so that $Y_q \to 0$ and then $S_q \to 0$ and $T_q \to 0$ also. If $S_q = \sum_i s_{iq} x_i$ and $T_q = \sum_i t_{iq} x_i$ then we let $R_q = \sum_i r_{iq} x_i$ be the element of $\mathfrak{g}$ whose coordinates r_{iq} satisfy

$$r_{iq} = r_i(s_{1q}, \cdots, s_{nq}, t_{1q}, \cdots, t_{nq}).$$

Since

$$\exp(R_q) = \exp(-S_q)\exp(Y_q),$$

$\exp(R_q) \in G_{m_q}$.

For sufficiently large q,

$$\begin{aligned} |r_{iq} - t_{iq}| \leqq \sum_j |r_{ij}(s_{1q}, \cdots, s_{nq}) - \delta_{ij}| \sup_j |t_{jq}| \\ + \sum_{j,k} |r_{ijk}(s_{1q}, \cdots, s_{nq}, t_{1q}, \cdots, t_{nq})| \sup_j |t_{jq}| \\ \leqq \left(\sup_j |t_{jq}|\right) \Big/ 3n. \end{aligned}$$

Thus

$$\|R_q - T_q\| \leqq \left(\sup_j |t_{jq}|\right) \Big/ 3 \leqq \|T_q\|/3,$$

where $\|Y\| = (\sum_i y_{i_q})^{1/2}$ if $Y = \sum_i y_i x_i \in \mathfrak{g}$, and so $2\|T_q\|/3 \leqq \|R_q\| \leqq 4\|T_q\|/3$. We write $R_q = P_q + Q_q$ where $P_q \in \mathfrak{g}_m{}^2$ and $Q_q \perp \mathfrak{g}_m{}^2$. Then $\|P_q\| \leqq \|R_q - T_q\| \leqq \|T_q\|/3$ since P_q is also the orthogonal projection of $R_q - T_q$ onto $\mathfrak{g}_{m_q}$. It follows that $\|Q_q\| \geqq \|T_q\|/3$.

Let $(\rho(q))^{-1}$ be the smallest integral power of 2 which is greater than $\|T_q\|$. Such a power exists since $g_q \notin G_{m_q 0}$, $Y_q \notin \mathfrak{g}_{m_q}$ and $T_q \neq 0$. Since $T_q \to 0$, $(\rho(q))^{-1} \to 0$ and $\rho(q) \to \infty$; also

$$1/2 \leqq \|\rho(q) T_q\| < 1,$$
$$1/3 \leqq \|\rho(q) R_q\| < 4/3.$$

Passing to a subsequence of the q's, we can suppose that $\rho(q)R_q$ converges to a nonzero element R of $\mathfrak{g}$. If δ is any diadic rational then for sufficiently large q, $\rho(q)\delta$ is an integer and

$$\exp(\delta R)m = \lim_q \exp(\delta\rho(q)R_q)m_q = \lim_q m_q = m,$$

since $\exp(R_q) \in G_{m_q}$. Thus $\exp(\xi R) \in G_{m_q}$ for each real ξ and so $R \in \mathfrak{g}_m$. However, it follows from the second paragraph of this proof that $\rho(q)Q_q$, the orthogonal projection of $\rho(q)R_q$ onto $\mathfrak{g}_{m_q}^{\perp}$, tends to the orthogonal projection of R onto $\mathfrak{g}_m^{\perp}$. This projection must be zero since $R \in \mathfrak{g}_m$ but it must be nonzero since

$$\|\rho(q)Q_q\| \geqq \|\rho(q)T_q\|/3 \geqq 1/6.$$

This is a contradiction and so the required neighborhood N of e must exist.

Proof of Lemma 4. (3)$\Rightarrow$(2): We assume the denial of (2). There is a g in G_m and a neighborhood N of e and we can find a sequence p_k in M converging to m such that $g \notin NG_{p_k}$ for $k=1, 2, \cdots$. Then $N^{-1}g \cap G_{p_k} = \emptyset$ for $k=1, 2, \cdots$ and so there cannot exist a subsequence $k(1)$, $k(2)$ of the integers and $g_{k(i)}$ in $G_{p_{k(i)}}$ such that $g_k \rightarrow g$. This proves the denial of (3).

(2)$\Rightarrow$(1): We assume the denial of (1). There is a g in G_m, a neighborhood N of e, a sequence m_k in M converging to m and an infinite set K of integers such that for k in K, $Ng \cap G_{m_k} = \emptyset$. This implies $g \notin N^{-1}G_{m_k}$ for k in K and so there is no neighborhood U of m such that for p in U, $g \in N^{-1}G_p$ and this proves the denial of (2).

(1)$\Rightarrow$(3): This is obvious.

This completes the proof of Theorem 3.

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